Preface to the Tarquin Edition

I appreciate the initiative and skill of Kate Crennell* in undertaking a new edition of our old treatise. I take this opportunity to explain how the first edition came into being and to add a few words about the tragic life and death of my school friend, John Flinders Petrie.

In the early nineteen-twenties, Petrie and I happened to be incarcerated in the same boarding school. At that stage, my chief interest was in composing music, and it may well have been through his influence that I abandoned musical composition in favour of the geometry of polyhedra and their analogues in higher dimensions. Petrie had an uncanny ability to "visualize" 4-dimensional space. Sometimes, when I asked him a question about hyperspace, he would close his eyes, concentrate, and eventually tell me the answer.

About 1932, when I had been elected a Fellow of Trinity College, Cambridge, Professor J.E. Littlewood asked me to assess a collection of polyhedral models which the crystallographer H.T. Flather had offered to donate to the University. So I drove over to St. Albans, met him and his sister and returned to Cambridge with my favourable report. His small stature prevented me from understanding that he was my senior by some thirty years. When I showed him that his collection of stellated icosahedra was incomplete, he soon made exquisite models of the rest.

Patrick Du Val, who wrote the Preface to the 2nd edition, was another Fellow of Trinity College, my senior by a few years. He and I attended many lectures together, including a course given by Littlewood on "The Theory of Real Functions." Littlewood's special method was to publish a small book on that subject and to give us copies to keep on our desks. Each lecture was a commentary on one or two pages of that booklet. When I asked DuVal what he thought of it, he remarked that the first letter of the Hebrew alphabet (א - aleph) was printed upside down in every occurrence except one. He taught me to appreciate the role of quaternions in pure geometry and he corroborated my enumeration of the 59 icosahedra by discovering a new method. He was the author of several other books including "Elliptic Functions and Elliptic Curves." He and I remained good friends for the rest of his life.

* See note on production of this edition on the previous page

Our symbols		Wheeler	Brückner
A	(the icosahedron itself)	1	(not shown)
B	(a "triakisicosahedron")	2	Fig. 2, Taf. VIII
C	(the five octahedra)	3	Fig. 6, Taf. IX
D		4	Fig.17, Taf. IX
$\mathrm{De_2}$		5	
E$\boldsymbol{\mathit{f}_1}$	(the five tetrahedra, *laevo*)	6	(not shown)
E$\mathbf{f_1}$	(the five tetrahedra, *dextro*)	7	Fig. 11, Taf. IX
$\mathrm{Ef_1}$	(the ten tetrahedra)	8	Fig. 3, Taf. IX
$\mathrm{Ef_1g_1}$		9	Fig. 26, Taf. VIII
$\mathrm{De_1}$		10	
G	(the great icosahedron)	11	Fig. 24, Taf. XI
H	(the "complete" stellation)	12	Fig. 14, Taf. XI
$\mathrm{De_2f_2}$		13	
$\mathrm{Fg_2}$		14	
E$\mathbf{f_1}\mathrm{f_2}$	(*dextro*)	15	
E$\boldsymbol{\mathit{f}_1}\mathrm{f_2}$	(*laevo*)	16	
$\mathrm{Fg_1}$		17	Fig. 3, Taf. X
$\mathrm{Ef_2}$		18	Fig. 20, Taf. IX
F		19	
$\mathrm{e_1f_1g_1}$		20	
$\mathrm{g_1}$		21	
$\mathrm{f_2}$		22	

As might be expected in such a case, an exhaustive enumeration greatly increases the list. We find, altogether, thirty-two reflexible icosahedra (including the ordinary and "great" icosahedra, and the compounds of five octahedra and of ten tetrahedra) and twenty-seven enantiomorphous pairs (including the pair of compounds of five tetrahedra, and Wheeler's Nos. 15, 16). Of the latter, we would draw special attention to the skeletal pair $\mathrm{f_1}$, f_1, either of which may be described as that part of one set of five tetrahedra which is exterior to the other set. For, it will appear that every enantiomorphous pair can be derived from one or two reflexible icosahedra by adding $\mathrm{f_1}$ and f_1, in turn.

The point X would represent the tetrad $(5,\ \mathit{5},\ 10,\ \mathit{10})$.) The possible closed chains are:

$$\lambda\ 6\ \mu\ \mathit{6},\qquad \lambda\ 6\ \mathit{9}\ \nu\ 9\ \mathit{6},\qquad \mu\ \mathit{9}\ \nu\ 9;$$

the possible open chains are:

$$\mathit{5}\ \lambda\ 5,\qquad \mathit{5\ 6}\ \mu\ 6\ 5,\qquad \mathit{5\ 6}\ 9\ \nu\ \mathit{9}\ 6\ 5,$$
$$10\ \nu\ \mathit{10},\qquad 10\ 9\ \mu\ \mathit{9\ 10},\qquad 10\ 9\ \mathit{6}\ \lambda\ 6\ \mathit{9\ 10},$$
$$\mathit{5\ 6}\ 9\ 10,$$
$$5\ \lambda\ \mathit{6}\ 9\ 10,\qquad 5\ 6\ \mu\ 9\ 10,\qquad 5\ 6\ \mathit{9}\ \nu\ 10,$$
$$\mathit{5}\ \lambda\ 6\ \mu\ 9\ 10,\qquad \mathit{5}\ \lambda\ 6\ \mathit{9}\ \nu\ 10,\qquad \mathit{5\ 6}\ \mu\ \mathit{9}\ \nu\ 10,$$
$$5\ \lambda\ \mathit{6}\ \mu\ \mathit{9}\ \nu\ 10$$

and eight others which can be derived from the last eight of these by transposing the Roman and italic numerals.

In all cases, the conditions imposed by the tetrads:

$$(5, \mathit{5}, 9, \mathit{9}),\qquad (5, \mathit{5}, 10, \mathit{10}),\qquad (6, \mathit{6}, 10, \mathit{10})$$

are automatically satisfied. Remembering that λ stands for **3** or **4**, μ for **7** or **8** and ν for **11** or **12**, we now have thirty-one reflexible icosahedra, whose faces are:

$$\mathbf{0},\ \ \mathbf{1},\ \ \mathbf{2},\ \ \mathbf{13},\ \ \mathbf{3\,4},\ \ \mathbf{7\,8},\ \ \mathbf{11\,12},$$
$$\lambda\ 6\ \mathit{6}\ \mu,\qquad \lambda\ 6\ \mathit{6}\ 9\ \mathit{9}\ \nu,\qquad \mu\ 9\ \mathit{9}\ \nu,$$
$$\lambda\ 5\ \mathit{5},\qquad 5\ \mathit{5}\ 6\ \mathit{6}\ \mu,\qquad 5\ \mathit{5}\ 6\ \mathit{6}\ 9\ \mathit{9}\ \nu,$$
$$10\ \mathit{10}\ \nu,\qquad \mu\ 9\ \mathit{9}\ 10\ \mathit{10},\qquad \lambda\ 6\ \mathit{6}\ 9\ \mathit{9}\ 10\ \mathit{10}$$

and twenty-seven enantiomorphous pairs, whose faces (one from each pair) are:

$$\mathit{5\ 6}\ 9\ 10,$$
$$\lambda\ 5\ \mathit{6}\ 9\ 10,\qquad 5\ 6\ \mu\ 9\ 10,\qquad 5\ 6\ \mathit{9}\ 10\ \nu,$$
$$\lambda\ \mathit{5}\ 6\ \mu\ 9\ 10,\qquad \lambda\ \mathit{5}\ 6\ \mathit{9}\ 10\ \nu,\qquad \mathit{5}\ 6\ \mu\ \mathit{9}\ 10\ \nu,$$
$$\lambda\ 5\ \mathit{6}\ \mu\ \mathit{9}\ 10\ \nu.$$

The enantiomorphous pair $\mathit{5\ 6}\ 9\ 10$, $5\ 6\ \mathit{9\ 10}$ is peculiar in having no common part. Accordingly, we add the combination:

$$5\ \mathit{5}\ 6\ \mathit{6}\ 9\ \mathit{9}\ 10\ \mathit{10}$$

to the list of reflexibles. The enumeration is now complete.

It is convenient to simplify the symbols for the reflexible faces by writing **5** for $5\ \mathit{5}$, **6** for $6\ \mathit{6}$, **9** for $9\ \mathit{9}$ and **10** for $10\ \mathit{10}$. We may now let λ stand for any one of **3**, **4**, **5**, and ν for any one of **10**, **11**, **12**, so that the list of (thirty-two) reflexibles becomes:

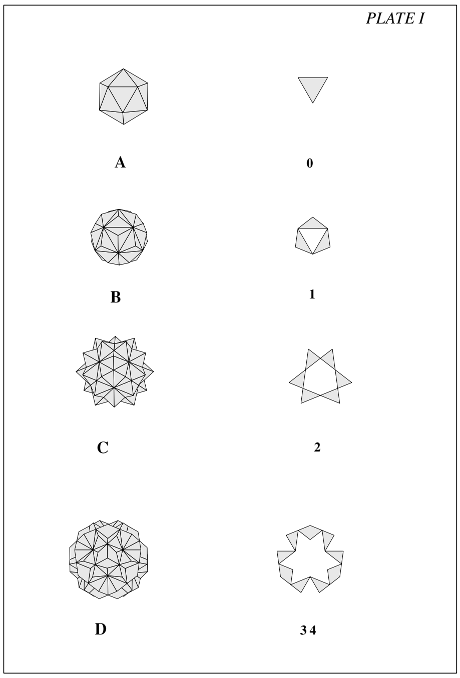
$$\mathbf{0},\ \mathbf{1},\ \mathbf{2},\ \mathbf{13},$$
$$\mathbf{3\,4},\ \ \mathbf{3\,5},\ \ \mathbf{4\,5},\ \ \mathbf{7\,8},\ \ \mathbf{10\,11},\ \ \mathbf{10\,12},\ \ \mathbf{11\,12},$$
$$\lambda\ \mathbf{6}\ \mu,\qquad \mu\ \mathbf{9}\ \nu,$$
$$\lambda\ \mathbf{6\ 9}\ \nu.$$

(For the chiral figures, we must let λ and ν retain their original meaning.)

One symmetry of the graph (Fig. 5) exhibits the duality between enantiomorphous pairs of solids. Another exhibits the possibility of subtracting every number from 15; but this duality has no obvious geometrical significance. (We may regard **2** as being paired with **13**, but **0** and **1** are exceptional in having no companions.) So far, we have made no distinction between "positive" and "negative" regions of a face, *i.e.*, between regions where the inside ("substance") of the solid is *below* the plane of the face (regarded as lying above the centre) and regions where the inside is *above*. Hereafter, we shall distinguish a "negative" region by affixing a dash (') to the number; *e.g.*, we shall write **3' 5** instead of **3 5**. In the Plates, each solid is drawn on the left, and its face on the right (with "negative" regions having dark shading).

3. An Alternative Enumeration, by Considering Solid Cells

A somewhat different notation for the icosahedra, with perhaps a clearer idea of their character, can be obtained by considering, instead of the regions into which each plane is divided by the traces on it of the others, the analogous three-dimensional regions, or *cells*, into which space is divided by the whole set of twenty planes. We shall employ one clarendon symbol for a whole set of cells permuted into each other by the extended icosahedral group. Such a set may consist of 12, 20, 30, 60 or 120 cells, according to the symmetry of the individual cell; in the last case, the cell has no symmetry at all, and the set consists of two enantiomorphous sub-sets of 60 each, which will be indicated by corresponding Roman and italic symbols, as in the case of the plane regions. (This is found to occur in the case of one set of cells only, that forming the solid whose face-symbol is **5' 6' 9 10**; in all other cases the single region is reflexible.)



In the remaining plates a darker shading is applied to the negative or under sides of the planes.

PLATE IV.

$\mathbf{e_1}$ consists of a set of 20 spikes with blunt bases, triangular below and fitting into the triangular depressions of **D**; and ditrigonal above, *i.e.*, the section is a hexagon whose sides are equal and whose angles have two values alternately, greater and less respectively than the angle of a regular hexagon. The set of spikes is vertex-connected, *i.e.*, each has a vertex (but no edge) in common with each of three others. $\mathbf{f_1}$ consists of 120 scalene tetrahedra, and is the only one of the layers and partial layers whose individual pieces have no symmetry at all; 60 of them are of course the mirror images of the other 60. They fit together in cycles of six (three of each kind arranged alternately) to make 20 pyramidal sheaths, fitting over the spikes $\mathbf{e_1}$ and converting them into double pyramidal spikes, whose section consists of two crossed triangles, like the face of the five octahedra **C**. These sheaths meet by pairs in a couple of edges, so that the figure is edge-connected throughout, forming a shell with twelve decagonal holes in it. Each double triangular spike has three wider and three narrower grooves down it, the former ending at the holes, and the latter meeting by pairs in 30 rhombic depressions. $\mathbf{g_1}$ consists of a set of narrow wedges which fit into these rhombic depressions, and whose acute edges form the edges of a large dodecahedron, with its vertices at the tips of the spikes $\mathbf{e_1}$.

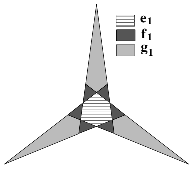


Fig. 6

The accompanying sketch (Fig. 6) of the section of $\mathbf{e_1}$, $\mathbf{f_1}$, $\mathbf{g_1}$ by a plane perpendicular to a circumradius of this dodecahedron, and near its vertex, makes clear the relation between them.

PLATE VI.

$\mathbf{f_2}$ is the only one of the figures whose pieces are not even vertex connected. Each of the 12 is a long pentagonal spike with a comparatively blunt pentagonal base; the two pentagonal pyramids (acute and obtuse) by which it is bounded being oppositely placed, so that the edges of the one meet the faces of the other, and the lateral edges of each spike are a skew decagon, which just fits into one of the decagonal holes of $\mathbf{f_1}$, or of any of the figures in Plate V. The obtuse (inner) pyramid of each piece is the solid angle vertically opposite to that of the original icosahedron **A** at a vertex. $\mathbf{e_2}$ and $\mathbf{g_2}$, though very dissimilar in appearance, are descriptively much alike; each is partly edge-connected and partly vertex-connected, the pieces fitting together by fives along edges to form 12 pyramidal sheaths, which fit onto one set of pyramids of $\mathbf{f_2}$, converting them from pentagonal to pentagrammatic pyramids. This is made clear by the accompanying sketch (Fig. 7) of a section of $\mathbf{f_2}$ and either $\mathbf{g_2}$ or $\mathbf{e_2}$ by a plane perpendicular to a pentagonal axis, and near a vertex of $\mathbf{f_2}$. The chief differences between $\mathbf{e_2}$ and $\mathbf{g_2}$ are the much greater height of each pyramidal sheath in the latter in proportion to its base, and the fact that the pyramids in the former are pointed inwards towards the centre of the figure, in the latter outwards. The 12 sheaths in each case are connected by vertices. It may be noted that the pentagrammatic under sides of $\mathbf{e_2}$ fit into the pentagrammatic depressions of **D**, converting them into pentagonal depressions. The five pieces of $\mathbf{e_2}$ at a pentagonal vertex occupy the solid angles vertically opposite to the five pieces of **B** at that vertex.

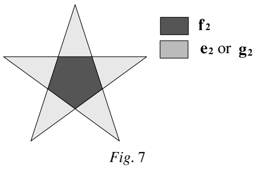


Fig. 7

The interstices through which it is possible to see the inside of $\mathbf{g_2}$ are so narrow that it has been thought best to draw the figure in section, only (the farthest) 6 of the 12 sheaths appearing.

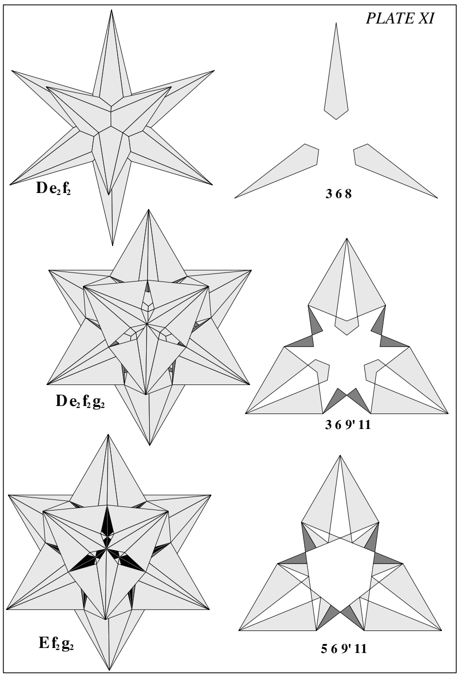
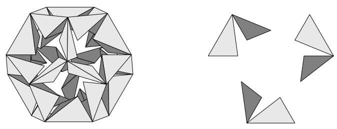
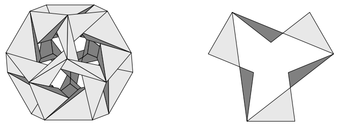


PLATE XIII



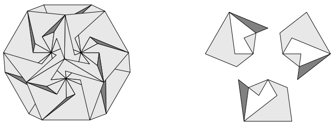
$\mathrm{f}_1\mathbf{g}_1$

5′ 6′ 9 *10′* **12**



$\mathbf{e}_1\mathrm{f}_1\mathbf{g}_1$

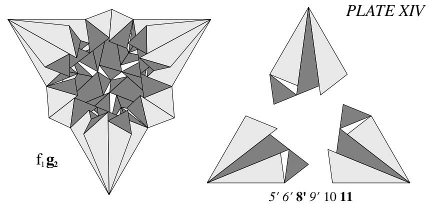
3′ 5 *6′* 9 *10′* **12**



$\mathbf{D}\mathbf{e}_1\mathrm{f}_1\mathbf{g}_1$

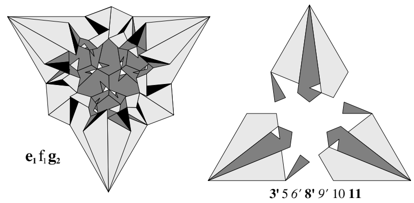
4 5 *6′* 9 *10′* **12**

PLATE XIV



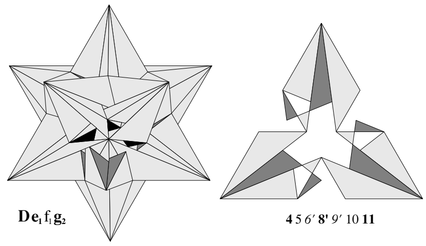
$\mathrm{f}_1\mathbf{g}_2$

5′ 6′ **8′** *9′* 10 **11**



$\mathbf{e}_1\mathrm{f}_1\mathbf{g}_2$

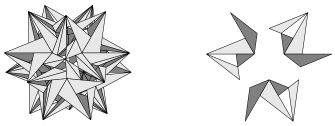
3′ 5 *6′* **8′** *9′* 10 **11**



$\mathbf{D}\mathbf{e}_1\mathrm{f}_1\mathbf{g}_2$

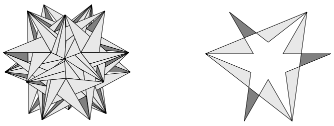
4 5 *6′* **8′** *9′* 10 **11**

PLATE XVI



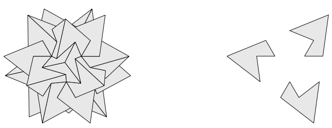
$\mathbf{e}_2\mathrm{f}_1$

4′ *5′* 6 **7** 9 10



$\mathbf{D}\mathbf{e}_2\mathrm{f}_1$

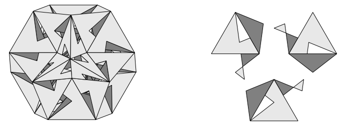
3 *5′* 6 **7** 9 10



$\mathbf{E}\mathrm{f}_1$

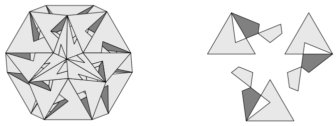
5 6 **7** 9 10

PLATE XVII



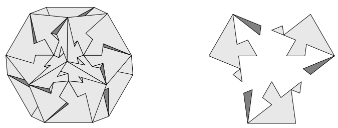
$\mathbf{e}_2\mathrm{f}_1\mathbf{g}_1$

4′ *5′* 6 **7** 9 *10′* **12**



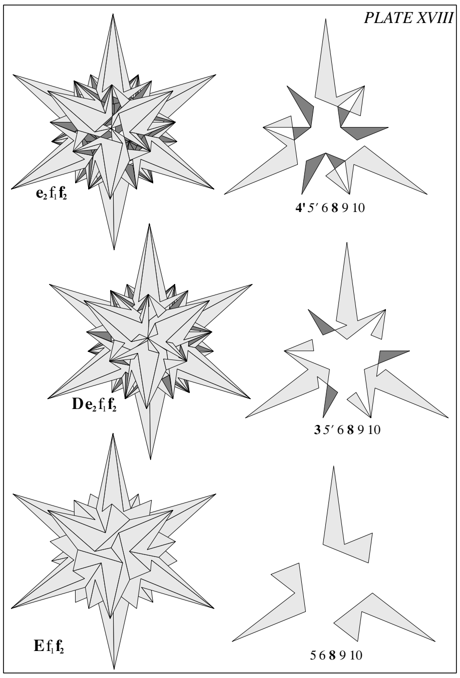
$\mathbf{D}\mathbf{e}_2\mathrm{f}_1\mathbf{g}_1$

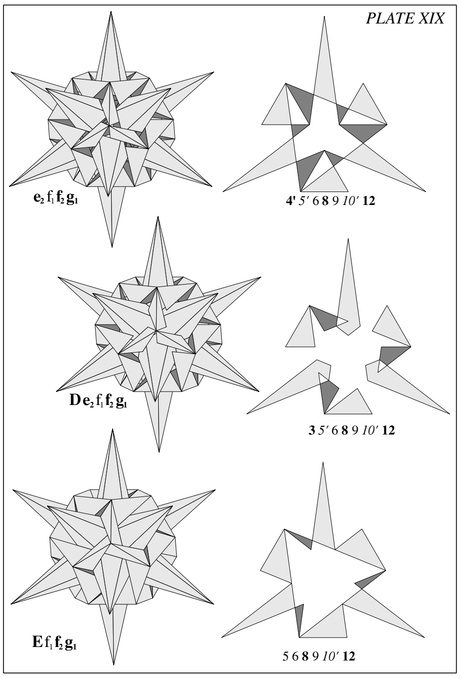
3 *5′* 6 **7** 9 *10′* **12**



$\mathbf{E}\mathrm{f}_1\mathbf{g}_1$

5 6 **7** 9 *10′* **12**





The models were mostly made from thin white card; some like model 33 opposite were coloured. Each photograph is labelled with the model number, Du Val symbol, Plate number and *colour* (in italic text). A few models, such as number 16 (shown overleaf), were held together with wire. Most of the models were rigid structures but a few, such as number 33 (opposite) are fragile since they were glued only at the corners.

photo unchanged

model 55 : $\mathbf{De_2} f_1 \mathbf{f_2 g_1}$ (plate XIX) *white*